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Article

Brownian Motion in Minkowski Space

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Abstract: We construct a model of Brownian motion in Minkowski space. There are two aspects of the problem. The first is to define a sequence of stopping times associated with the Brownian "kicks" or impulses. The second is to define the dynamics of the particle along geodesics in between the Brownian kicks. When these two aspects are taken together, the Central Limit Theorem (CLT) leads to temperature dependent four dimensional distributions defined on Minkowski space, for distances and 4-velocities. In particular, our processes are characterized by two independent time variables defined with respect to the laboratory frame: a discrete one corresponding to the stopping times when the impulses take place and a continuous one corresponding to the geodesic motion in-between impulses. The subsequent distributions are solutions of a (covariant) pseudo-diffusion equation which involves derivatives with respect to both time variables, rather than solutions of the telegraph equation which has a single time variable. This approach simplifies some of the known problems in this context.

Keywords: geodesic; quaternions; stopping times; Markov processes; pseudo-diffusion equation

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1. Introduction

Brownian motion is one of the cornerstones of statistical mechanics. Einstein successfully used it to give a rational proof of the existence of atoms [1,2], and since then it has been used as a paradigm to model systems in contact with a heat reservoir. It has also been used as a stochastic model to represent a variety of different phenomena in such diverse fields as physics, chemistry, biology, finance *etc.* Indeed, its universal character rests on it being the simplest model available for describing time evolution implied by a combination of random and deterministic factors [3–5]. In the case of Einstein's theory, the deterministic factor is given by the Stokes force exerted on pollen grains by a liquid, seen as a continuum macroscopic medium; while the random factor represents the impulses given to the same pollen grains by the myriad of fluid molecules colliding with them. Einstein's ingenuity consisted in understanding that water could be seen as acting on pollen grains in two almost antithetical ways: as a continuum with its viscosity (systematic component), and as a collection of many interacting particles (chance).

Many phenomena can be interpreted as the result of the cooperation of systematic and random events, hence the success of this simple model in combining the two. For example, in the case of a tagged molecule of a gas, the free flights between collisions with other molecules constitute the systematic part, while the chance collisions with other molecules which interrupt the free flights, constitute the non-systematic component of the motion causing it to move in a random environment [6,7]. In principle, this picture applies to all forms of dynamics, including special and general relativity, provided a universal time parameter is defined. In the case of relativity, this universal time was first introduced by Stueckelberg [8,9] and later developed by Oron and Horwitz to define Relativistic Brownian Motion [10]. The results of this paper have some similarity with theirs, although the approach is quite different.

We can, therefore, explore the possibility of addressing the relativistic Brownian motion, as a random process with stationary independent increments, in which a Brownian particle travels on a geodesic until it is dislodged by the interaction with another (moving or standing) object, which shifts it onto a new geodesic. Seen from this perspective, there are two aspects to consider. One is connected with the specific dynamics of the particle along the smooth parts of its piecewise smooth trajectory, while the second is connected with the random fluctuations that occur as the particle bounces from one smooth section of the trajectory to another. In practice, the geodesic motion between the interactions obeys the deterministic laws of relativistic dynamics, while the collection of impulses assigns random orientations, positions, velocities and accelerations according to appropriate statistical laws. We will focus on two of these statistical processes. One associated with particle position which will give a four dimensional distribution defined on Minkowski space, the other associated with the 4-velocities. In this latter case the velocity along geodesics will be a consequence of the proper time [11] being a function of a universal time [12].

To fully implement this, it is necessary to separate the "random" parts from the deterministic parts of the motion [13]. We do this by means of stopping times and then, in order to be coherent, we refer all motions and stopping times to the same (laboratory) frame of reference by means of a universal time parameter. The use of this parameter essentially transposes a three dimensional classical problem, with an absolute time τ, into a four dimensional problem on Minkowski space, with τ now serving as a universal (although not absolute) time parameter. Two cases arise:

(1) A Brownian random walk in which a single time variable determines both the stopping times and the time between jumps. For example, one could adapt Keller's analysis [14] of the model of a particle moving along the x axis such that during a time interval of duration $\Delta\tau = 1$ from $\tau = i - 1$ to $\tau = i$, $i = 1, 2, \ldots$, the particle moves with velocity $\nu = +1$ or $\nu = -1$, each with probability $1/2$.

(2) A Brownian random walk in which two independent time variables are used, the stopping times and the the proper time difference between two consecutive stopping times, both of which are measured with respect to the universal time τ.

For the purpose of this article, we focus on case (2). We shall assume that the second time variable defines a stochastic process which obeys the Strong Markov property [15] indexed by the stopping times, with the Brownian flights lying along a piecewise differentiable curve of random length each with its own proper time. We propose a new perspective, based on combining the discrete stopping times and the continuous time associated with the deterministic part of the motion. The proper time intervals between two events are invariant by definition, while multiple events with respect to the same frame can be time ordered by means of a universal time parameter [10,16]. Indeed, from the perspective of the rest frame of the particle, Brownian impulses recorded at proper times $\hat{\tau}_1, \hat{\tau}_2, \ldots \hat{\tau}_i, \ldots$, where each $\hat{\tau}_i$ is a function of the universal time τ, can be transmitted to the laboratory frame, the frame in which the heat bath can be considered at rest.

For $\hat{\tau} \in [\hat{\tau}_{i-1}, \hat{\tau}_i)$ let $\chi(\hat{\tau}) = \hat{\tau} - \hat{\tau}_{i-1}$ be the proper time between two Brownian impulses, and $s(\hat{\tau}) = c\chi(\hat{\tau})$, where c is the speed of light. When $\hat{\tau} = \hat{\tau}_i$, let the numbers $\chi_i = \hat{\tau}_i - \hat{\tau}_{i-1}$ and $s_i = c\chi_i$ define two random variables χ and s, that represent the times between successive impulses as recorded in the rest frame and the corresponding length between impulses respectively. Their variances, Var(χ) and Var(s) $= c^2 Var(\chi)$, depend on properties of the bath, such as its density ϱ and temperature Θ, which we assume to be constant; hence, for simplicity, we omit these parameters in our notation. Also, we assume Var(χ) is positive and therefore that the trajectory segments between impulses are timelike.

As each time increment χ_i is a proper time (recall the local time of the rest frame is equivalent to the proper time) it remains invariant in general relativity under coordinate transformations, which also includes the laboratory frame. This is the only information transmitted from frame to frame. Moreover, as the heat bath is held at a constant temperature, then it is reasonable to assume using the strong Markov property that the set $\{\chi_i\}_{i=1}^{\infty}$ defines a set of stationary independent increments [17]. This statement is independent of reference frames, and will be assumed throughout.

It is also true that from the laboratory frame's perspective we do not know the local coordinates (t, x, y, z) of the Brownian particle and that many options are possible. However, we can extract enough information from the invariance of the set $\{\chi_i\}_{i=1}^{\infty}$ and calculate the probability of finding the particle within the range of applicability of the Central Limit Theorem (CLT) [18]. This range is asymptotically of the order of the standard deviation σ_χ of χ, and it contains most of the probability. Our approximation of the density function will be Lorentz invariant (see below) and therefore will allow us to calculate probabilities in any Lorentz frame.

To conclude, we will show how a relativistic Brownian motion can be described when motion between the discrete random stopping times lies along a geodesic path. In doing so, within the range of applicability, we obtain a covariant description of the relativistic Brownian motion, as e.g., in [19], and in the case of velocities we obtain a distribution which may be considered in place of the Maxwell-Jüttner distribution, e.g., in [20,21]. Both cases presuppose the use of two independent time parameters, one discrete and the other continuous. The novelty of our article is that relativistic Brownian motion can be described by pseudo-diffusion in Minkowski space, as opposed to, e.g., telegraph transport in 3-dimensional space [5,22].

Unless one is interested in a detailed description of the interaction processes [20,23,24], this approach constitutes one way to overcome some of the historical difficulties associated with relating the Brownian Motion and a relativistic Markov process by focusing on local time rather than proper time [5,25–27].

2. Geodesics

"According to the Principle of Equivalence," to quote Weinberg (p.70 in [28]) adjusted to our notation, "there is a freely falling coordinate system x^a in which its equation of motion is that of a straight line in space-time, that is

$$\frac{d^2x^a}{ds^2} = 0$$

with ds/c the proper time"

$$ds^2 = g_{ij}dx^i dx^j \tag{1}$$

This can also be written as

$$ds = g_{ij}\frac{dx^i}{ds}dx^j \ .$$

For much of the analysis to follow, it will be more convenient to describe the Brownian motion from the perspective of a single laboratory frame. Consequently, it will be important to not only associate different proper times with each component of the paths connecting stopping times but also to choose a common parametrization for all the curves (more details on curve parameters can be found in the Appendix at the end). We will refer to this as the universal time parameter τ. This is equivalent to the universal time parameter [29], first introduced by Stuekelberg in 1941 [8,9], and further developed by Horwitz and Piron [30]. It could be chosen to be the proper time of a standard clock at rest in the laboratory frame. It could also be taken as the time in the rest frame of the particle, as described in the last section. Indeed, if both parameters are connected by an affine parameter then the same time scale can be applied to both. However, in the case of non-affine parameters we have to allow for accelerations in the system (for an affine parameter there is no acceleration which means $\ddot{s} = 0$). Regardless of the choice of parameter, the random variable $c\chi_i = s_i$ will be invariant under parameter and coordinate transformations. In the case of a piecewise geodesic on the i-th component of the curve, we let

$$ds \equiv ds_i \ .$$

Moreover, each component s_i can be synchronized with the laboratory frame by means of the expression

$$ds_i = \frac{ds_i}{d\tau}d\tau \ .$$

where τ is the universal time parameter. Denoting $\dot{s} = ds/d\tau$, Equation (1) can be re-written as

$$\dot{s}ds = c^2\dot{t}dt - \dot{x}dx - \dot{y}dy - \dot{z}dz \ . \tag{2}$$

For what follows, we will restrict the motion to piecewise geodesic curves with Var$(\chi) > 0$; in other words, to those curves for which the tangent is defined uniquely at every point along a (smooth) piece of the trajectory in $\mathcal{M}$, in between two instantaneous random events occurring at discrete times $\hat{\tau}_{i-1}(\tau)$ and $\hat{\tau}_i(\tau)$. Here, each trajectory is a collection of random events mediated by the kicks which cause random deviations at random times in the free motion. These kicks may be due to collisions with heat bath particles or with various kinds of obstacles [6,31–33].

It follows that for each realization of the processes $\{\hat{\tau}_i\}_{i=1}^{\infty}$ defined with respect to the standard clock in the rest frame of the laboratory, $\{s_i\}_{i=1}^{\infty}$ and $\{\dot{s}_i\}_{i=1}^{\infty}$ define a Markov random walk with respect to the (well ordered) index set of stopping times $\hat{\tau}_i$, where $\dot{s}_i = \dot{s}(\hat{\tau}_i)$ means that the left derivative is computed at $\hat{\tau} = \hat{\tau}_i$.

As usual in Brownian motion theory, we assume that the only relevant parameter of the heat bath is its temperature Θ. The above can then be considered to be stationary independent processes and the specific forms of the resulting Markov processes are determined by their Θ-dependent distributions. Indeed, Θ influences the time increments χ_i, and the measure of a universal unit of time [34].

In particular, $\{s_i\}_{i=1}^{\infty}$ and $\{\dot{s}_i\}_{i=1}^{\infty}$ will have Θ-dependent infinitely divisible distributions which can be associated with Levy processes, while the variables $s = \sum_{i=1}^{n} s_i$ and $\dot{s} = \sum_{i=1}^{n} \dot{s}_i$ will converge in distribution to Θ-dependent stable processes [35], for finite constant Θ.

We assume (as suggested also by molecular dynamics studies such as [25,26]) that the underlying thermal bath state is characterized by the isotropy of test particle trajectories associated with the independent and identically distributed (with Θ-dependent) *time-like* increments χ_i of the stopping times (in this paper we do not investigate how the distribution of the increments χ_i depends on Θ, as we keep Θ fixed.)

3. Brownian Motion from the Perspective of the Laboratory Frame

Consider the complex random 4-vector $\mathbf{X}$ indexed by the stopping times, representing the increments $\mathbf{X}_j = (cT_j, iX_j, iY_j, iZ_j)$ between two impulsive events, such that [36]:

$$s_j^2 = \langle \mathbf{X}_j, \mathbf{X}_j \rangle = c^2 T_j^2 - X_j^2 - Y_j^2 - Z_j^2 \geq 0\,. \tag{3}$$

In the case of isotropy for the positions (X, Y, Z) and for the velocities (V_x, V_y, V_z), with $V_x^2 + V_y^2 + V_z^2 < c^2$, we assume $\mathbb{E}(X) = \mathbb{E}(Y) = \mathbb{E}(Z) = 0$, and $\mathbb{E}(V_x) = \mathbb{E}(V_y) = \mathbb{E}(V_z) = 0$, where $\mathbb{E}$ denotes expected value. The time elapsed between two impulsive events will be positive with mean $\mu_T > 0$ if we consider future events, but negative with mean $\mu_T < 0$ if we consider the past. Therefore, we may take $\mu_T = 0$, there is no loss of generality due to this assumption, as $\mu_T \neq 0$ merely implies a translation along the time axis. We further assume a positive standard deviation $\sigma_{cT} > 0$ and $\sigma_X = \sigma_Y = \sigma_Z = \sigma > 0$, where the equalities are due to the assumed isotropy (the isotropy condition can be easily relaxed, allowing different standard deviations for the different directions of space. However, we prefer to keep our notation simple). It follows from Equation (3) that the expected value given by

$$\mathbb{E}(s^2) = \mathbb{E}(T^2) - \mathbb{E}(X^2) - \mathbb{E}(Y^2) - \mathbb{E}(Z^2)$$

implies

$$\sigma_s^2 = \sigma_{cT}^2 - \sigma_X^2 - \sigma_Y^2 - \sigma_Z^2 = \sigma_{cT}^2 \left(1 - \frac{3\sigma^2}{\sigma_{cT}^2}\right) \text{ and } \sigma^2 \leq \frac{\sigma_{cT}^2}{3}\,.$$

Also the time-like condition implies that T, X, Y and Z are not independent random variables. Nevertheless, they can be considered independent far from the light cone surface, where

$$(cT)^2 \gg X^2 + Y^2 + Z^2 \tag{4}$$

and the constraint (3) is only weakly perceived. As a matter of fact, given an initial distribution of massive particles near the origin of Minkowski space, the probability of one such particle remaining close to the light cone for a long time is small. Indeed, collisions are the characteristic feature of the BM, and in that case they are more likely to slow down the particle than to preserve or increase its speed. Therefore, condition (4) should be better and better verified for larger and larger τ. This assumed independence extends *a fortiori* to the four components of the sum of the first n trajectory segments of a Brownian particle:

$$S_n = \sum_{i=1}^{n} \mathbf{X}_i \tag{5}$$

with X_i corresponding to the impulse at time $\hat{\tau}_i$. As the only constraint that our variables have to obey is the time-like condition, we also assume the X_i and their components are i.i.d. variables, in accord with the notion of the Brownian motion.

By the Central Limit Theorem (CLT) the joint distribution for the complex number components of S_n is approximated at large n by the normal distribution in Minkowski space, (See Appendix 2 for a justification of this form of CLT) with variance growing linearly with n, and density function given by

$$f_{S_n}(ct, ix, iy, iz; n) \sim \frac{1}{4\pi^2 n^2 \sigma_{cT} \sigma^3} \exp\left\{-\frac{c^2t^2}{2n\sigma_{cT}^2} + \frac{x^2+y^2+z^2}{2n\sigma^2}\right\} = \frac{e^{-\langle q,q\rangle}}{4\pi^2 n^2 \sqrt{|\Sigma|}} \tag{6}$$

where $|\Sigma|$ is the determinant of the covariance matrix, i is the imaginary unit, and

$$q = \frac{1}{\sqrt{2n}}\left(\frac{ct}{\sigma_{cT}}, i\frac{x}{\sigma}, i\frac{y}{\sigma}, i\frac{z}{\sigma}\right)$$

The integration to compute probabilities is with respect to the volume element $dt(idx)(idy)(idz)$ and not $dtdxdydz$. This avoids issues of divergence and defines a probability measure in Minkowski space. It should also be clear that $\tilde{s}^2 \equiv \langle q, q\rangle$ is invariant. This follows by noting that $\tilde{s}^2 = \Sigma_{ij}^{-1} x^i x^j$ expressed in tensor notation, and that for the Lorentz transformation $A = a_j^i$, $(x')^i = a_j^i x^j$, and

$$\tilde{s}^2 = ((\Sigma')^{-1})_{ij}(x')^i(x')^j = a_i^k a_j^l (\Sigma^{-1})_{kl} a_q^i x^q a_s^j x^s = (\Sigma^{-1})_{ij} x^i x^j \ .$$

In addition, the presence of the $|\Sigma|$ term means f_{S_n} transforms as a tensor density under Lorentz transformations, and $f_{S_n} dt(idx)(idy)(idz)$ is invariant.

Due to the presence of the two independent time variables t and n, expressing respectively the random time between two Brownian impulses and the random number of impulses at a given universal time τ, f_{S_n} formally obeys a four dimensional pseudo-diffusion equation [37]:

$$\frac{\partial}{\partial n} f_{S_n} = \left[\frac{\sigma_{cT}^2}{2c^2}\frac{\partial^2}{\partial t^2} - \frac{\sigma^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right] f_{S_n} \tag{7}$$

where the operator within square brackets is the Laplacian in Minkowski space, and n is treated like a continuous variable representing the flow of time. This is indeed expected for a distribution arising from the CLT, with time variable n.

It should be noted that in the discrete case discussed in this paper, the interval between stopping times is a random variable which can be arbitrarily large. This means that even for finite n, the distance $\sqrt{x^2+y^2+z^2}$ traveled in 3-dimensional space can be arbitrarily large, which differs from the case in which n is standard time, and the Brownian particle moves with finite speed.

Finally, it is important to recall that the Gaussian approximation implied by the CLT for large n restricts the range of validity of this distribution, and its corresponding pseudo-diffusion equation, to a timelike region of order $O(\sqrt{n}\,\ell)$, where ℓ is the mean space-time distance traveled between two Brownian impulses.

By a similar reasoning, one obtains a normal distribution for the momentum 4-vector

$$U_n = \frac{1}{n}\sum_{j=1}^{n}\begin{pmatrix} E_j/c \\ ip_j^x \\ ip_j^y \\ ip_j^z \end{pmatrix} \tag{8}$$

which takes the form

$$f_U\left(\frac{E}{c}, ip^x, ip^y, ip^z; n\right) \sim \frac{n^2}{4\pi^2\sigma_E\sigma_p^3}\exp\left\{-\frac{n}{2}\left[\frac{E^2}{c^2\sigma_E^2} - \frac{p^{x2}+p^{y2}+p^{z2}}{\sigma_p^2}\right]\right\} \tag{9}$$

and may be considered as an alternative to the Maxwell-Jüttner distribution, that concerns the average momentum 4-vector, rather than the momentum 4-vector. Since this is comparable in structure to (6), it clearly satisfies a pseudo-differential equation with (E, ip^x, ip^y, ip^z) replacing (t, ix, iy, iz) in (7). Note, the distribution for the 4-vector (ct, x, y, z) is compatible but obtained without reference to the distribution for the 4-momentum; it is only based on the validity of the CLT for (ct, x, y, z).

4. Concluding Remarks

Our approach leads to a characterization of the equilibrium state of the Brownian motion in the framework of Special Relativity which, along the lines of classical statistical mechanics, requires almost no information about the details of the interactions among the objects of interest [38]. Nevertheless, these details are important and implicit in our formulation. In the first place they are the underlying cause of the unpredictable motions associated with the random interaction times and are constitutive of the phenomenon. In the second place, the statistics of these interactions depend on the characteristics of the heat bath, such as its density ϱ and its temperature Θ. As we consider them to be homogeneous in space and constant in time, we only need to know that they are consistent with the strong Markov property. Knowledge of these details is implicitly contained in the relation between the stopping times and the laboratory frame time, and should be made explicit to fully characterize the relativistic Brownian motion.

To compare our approach with others in the literature mentioned above, let us consider an equivalent interpretation of Equations (6) and (7), introducing the normalized variable $\overline{S}_n = S_n/n$. For large n and within the limit of applicability of CLT, $\overline{S}_n$ is distributed as:

$$f_{\overline{S}}(ct, ix, iy, iz; n) \sim \frac{n^2}{4\pi^2 \sigma_{cT} \sigma^3} \exp\left\{ -\frac{n}{2} \left(\frac{c^2 t^2}{\sigma_{cT}^2} - \frac{x^2 + y^2 + z^2}{\sigma^2} \right) \right\}. \tag{10}$$

because the standard deviations go as $\sigma_t/\sqrt{n}$ and $\sigma/\sqrt{n}$. Then $f_{\overline{S}}$ formally obeys the following equation:

$$\left[n^2 \frac{\partial}{\partial n} + \frac{\sigma_{cT}^2}{2c^2} \frac{\partial^2}{\partial t^2} \right] f_{\overline{S}_n} = \frac{\sigma^2}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f_{\overline{S}_n} \tag{11}$$

In the case when t and n are directly related to each other, Equation (11) plays a role analogous to that of the telegraph equation, which arises in various contexts concerning the Brownian motion. This is the case, for instance, when t and n are proportional to each other (e.g., for simple random walks in a classical framework or, more generally, when t is a monotonic function of n). In particular, the telegraph equation arises when considering the large deviations of the position of a symmetric random walk [14], where the telegraph equation constitutes a more accurate approximation than standard diffusion, not being restricted to describe the small fluctuations of the process. The telegraph equation is also commonly found in the treatment of the relativistic Brownian motion, because the fronts of its solutions propagate at finite speed, as prescribed by relativity, *cf.* the very informative review [5]. However, these fronts are singular and look unphysical for massive particles [21]. For instance, one may observe that the singularities emerge also in random walks in which the forward scattering probability is larger than the backward scattering probability [39]. Moreover, the telegraph equation describes a non-Markovian processes, in accord with results by Dudley [40] and Hakim [41], who proved that in Minkowski spacetime there are no Lorentz invariant continuous Markov processes whose typical paths have non-constant non-vanishing velocities.

To conclude, the main differences between our and other approaches is that we distinguish two time variables, which allow us to preserve the Markov property and the diffusion law in Minkowski space. We express the flow of time through the variable n, which grows with the universal time τ, thus preserving the sequence of events observed in the laboratory frame, although n is a random function of τ, and it is not related to the time between impulses t. Thanks to the additional time variable n, we can index 4-dimensional space-time events $X_n = (ct, ix, iy, iz)_n$, with $n \in \mathbb{N}$ a function of the proper laboratory time and not the local time variable t, which is part of the event. By doing this, we avoid the difficulties that arise (for example in the telegraph equation) from using only the single time variable t as an index for the 3-dimensional space events $(x, y, z)_t$. Because of the second order time derivative, events described by the telegraph equation do not enjoy the Markov property, in accordance with the aforementioned theorem of Hakim [41]. Moreover the solutions of the telegraph equation do not necessarily remain normalized and positive at all times [5].

We argue that our restriction to a part of the timelike sector of Minkowski space is not a serious limitation for massive particles [42]. Indeed, the relativistic nature of the process remains evident in our distributions and equations. The above suggests that at the core of the molecular chaos hypothesis in relativity are the stationary independent random events produced at stationary and independent random stopping times.

Appendix

A. Properties of Curves

In this appendix we introduce some notion that may be useful in the treatmet of the general relativity case. In principle every non-null regular curve γ can be parameterized by its proper time parameter $\hat{\tau} = s/c$ as $\gamma(s) = (t(s), x(s), y(s), z(s))$ such that $|\gamma'(s)| = 1$. Such a curve $\gamma(s)$ is called a unit speed curve. Also, in the rest frame, the standard clock will correspond to the proper times, which remain invariant under coordinate transformations. However, when a tetrad is erected and we attempt to compare two different line segments (geodesics), we are constrained to fix the coordinate system (t, x, y, z) and distinguish their different proper times. We can interpret this in one of two ways: either we have one tetrad and two proper times (one for each curve) or one proper time invariant with respect to two tetrads. It cannot be both.

For example, consider the two metric equations

$$ds^2 = c^2 dt^2 - dx^2 \qquad \text{and} \qquad ds'^2 = c^2 dt^2 - 2dx^2 \tag{12}$$

defined with respect to the coordinate axes (t, ix) in Minkowski space, where $i = \sqrt{-1}$. If we assume that

$$\frac{d^2 t}{ds^2} = \frac{d^2 x}{ds^2} = 0 \tag{13}$$

then from the perspective of the coordinate axes (t, ix) these define two different geodesics in the space, one parameterized by s and the other by $s' = s'(s)$. For instance, if $ct = s/\sqrt{2}$ and $x = is/\sqrt{2}$ then

$$ds'^2 = ds^2 - dx^2 = \frac{3}{2} ds^2 \,. \tag{14}$$

On the other hand, they could also be seen as the same line segment but defined with respect to two different (or re-scaled) coordinate axes. In this case, Equations (12) can be rewritten as

$$ds^2 = dt^2 - dx^2 \qquad \text{and} \qquad ds^2 = dt'^2 - dx'^2 \text{ respectively} \,,$$

with

$$dt = \frac{ds'}{ds} dt' \qquad \text{and} \qquad dx = \frac{1}{\sqrt{2}} \frac{ds'}{ds} dx' \,.$$

We interpret this to mean that both of them determine the same geodesic.

Finally, we note that if s' is a non-affine parameter of s then the geodesic equations, given in Equation (13), expressed in term of s' transform into

$$\frac{d^2 t}{ds'^2} = -\left(\frac{ds}{ds'}\right)^2 \left(\frac{dt}{ds'}\right) \frac{d^2 s'}{ds^2} \text{ and } \frac{d^2 x}{ds'^2} = -\left(\frac{ds}{ds'}\right)^2 \left(\frac{dx}{ds'}\right) \frac{d^2 s'}{ds^2} \,. \tag{15}$$

In general the expression

$$\frac{d^2 x^i}{ds'^2} = -\left(\frac{ds}{ds'}\right)^2 \left(\frac{dx^i}{ds'}\right) \frac{d^2 s'}{ds^2} \tag{16}$$

defines the acceleration along the geodesic with respect to the time parameter s' and by definition of a non-affine parameter both s and s' if suitably chosen can be written as functions of one another.

B. Central Limit Theorem in Complex Space

We begin by stating the standard form of the CLT defined over an n-dimensional complex space. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed with mean $\mu \in \mathcal{R}^k$ and covariance Σ, where Σ has finite entries then

$$\sqrt{n}(\overline{\mathbf{X}}_n - \mu) \to N(0, \Sigma)$$

in distribution. In particular if $\mathbf{X} = (x_1, x_2, \ldots, x_k)$ and each x_i is independent with $I\!\!E(x_i) = 0$ then

$$\Sigma = diag(\sigma_1^2, \ldots, \sigma_k^2) \; .$$

In terms of the moment generating function

$$M(\sqrt{n}(\overline{\mathbf{X}}_n - \mu) \to \exp\left(r^2 \sum \sigma_i^2/2\right)$$

Taking $k = 4$ and letting $x_1 = ct$, $x_2 = ix$, $x_3 = iy$ and $x_4 = iz$ we find using the standard 1-dimensional proof of the CLT that

$$M(\sqrt{n}(\overline{\mathbf{X}}_n - \mu) \to \exp\left((\sigma_{ct}^2 - \sigma_x^2 - \sigma_y^2 - \sigma_z^2)r^2/2\right)$$

which is clearly invariant under Lorentz transformations. It also follows that

$$f_{S_n}(ct, ix, iy, iz; n) \sim \frac{1}{4\pi^2 n^2 \sigma_{ct}\sigma^3} \exp\left\{-\frac{c^2t^2}{2n\sigma_t^2} + \frac{x^2 + y^2 + z^2}{2n\sigma^2}\right\} = \frac{e^{(-\langle q,q \rangle)}}{4\pi^2 n^2 \sqrt{|\Sigma|}}$$

which is Equation (6).

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Author Contributions

The authors have contributed equally to the formulation of the problem and to the calculations reported here. Both authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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7. Equivalent situations are realized with particles tracing deterministic trajectories in regular environments, if correlations decay in time and space, making inapplicable a deterministic description. This happens, for instance, in the so-called periodic Lorentz gas, consisting of point particles moving in a periodic array of convex (typically circular) scatterers, in which position and velocity correlations decay at an exponential rate, [31,32]. Another example is given by polygonal billiards, in which correlations do not decay exponentially fast [33]. In that case, one observes a different class of phenomena, which imply anomalous rather than standard diffusion.
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11. Proper time is given by s/c where $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ in the coordinate system (t, x, y, z) and is invariant under Lorentz transformations. Note that t corresponds to local time and should not be confused with the proper time.
12. In the event that the universal time is a non-affine parameter of the proper time the theory could also be extended to include accelerations. For an affine parameter there is no acceleration by definition.
13. This is often done. For instance, in order to formulate relativistically the quantum mechanical measurements, a piece of matter may be viewed as a "galaxy" of events, *i.e.*, of space-time points (called "flashes") at which the wave function collapses [43]. Flashes constitute the random part of the dynamics, while the unitary evolution of the wave function between flashes constitutes the systematic part. In the classical mechanics of particles, where there is an obvious choice for the universal time, one speaks of "event-driven" dynamics: practically random collisions (events) separate the (systematic) free flight evolutions.
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15. The Strong Markov property by definition is an extension of the standard Markov property (see e.g., [35] and [44] for a very recent study) to processes indexed by stopping times, also called optional times, see e.g., Sections 1.3 and 2.3 in [45].
16. Oron and Horwitz state "Brownian motion, thought of as a series of "jumps" of a particle along its path, necessarily involves an ordered sequence. In the nonrelativistic theory, this ordering is

naturally provided by the Newtonian time parameter. In a relativistic framework, the Einstein time t does not provide a suitable parameter. If we contemplate jumps in spacetime, to accomodate a covariant formulation, a possible spacelike interval between two jumps may appear in two orderings in different Lorentz frames. We therefore adopt the invariant parameter introduced by Stueckelberg in his construction of a relativistically covariant classical and quantum dynamics."

17. This may not be reasonable in General Relativity, since the density of the kicks may be affected by the gravitational field. For example, in the case of the Schwartschild metric, one might expect that for a closed system in equilibrium the Brownian kicks would have the same intensity on the hypersurface given by $r = h(t, \theta, \phi)$ where h is a given function and r is constant. However, for different values of r it will not be so. From the perspective of the heat bath it would mean that it is difficult to maintain a constant temperature except on the hypersurface. A more detailed discussion of Brownian Motion in the context of General Relativity can be found in [46].
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23. For instance, in the low density limit, in which the interaction (potential) energy is negligible compared to the kinetic energy, [25] describes a relativistic gas as a collection of particles which move according to special relativity from collision to collision, and treats as classical the "randomly" occurring collisions among particles. In this way [25] provides numerically a dynamical justification of the hypothesis of molecular chaos underlying the validity of the relativistic Boltzmann equation and of its equilibrium solution known as the Maxwell-Jüttner distribution [20,47].
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27. This has been investigated in great detail by Hakim [41,48] who defines relativistic stochastic processes in $\mu = \mathcal{M}^4 \times U^4$ where $\mathcal{M}^4$ is the Minkowski space-time and U^4 is the space of velocity 4-vectors but shows it is not suitable for defining a Markov process. Indeed, with the exception of a non-trivial time-discrete relativistic Markov model found in [21], certain relativistic generalizations and their Gaussian solutions must necessarily be non-Markovian or reduce to singular functions [21,41] (*cf.* the excellent Review [5], and references therein).
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34. The universal time is a standardized time defined within the space. It is not absolute time in the Newtonian sense.
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36. A Lorentz invariant inner product for two arbitrary vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in Minkoski space can be defined by
$$\left\langle \mathbf{x}_1, \mathbf{x}_2 \right\rangle \equiv= x_o y_o - x_1 y_1 - x_2 y_2 - x_3 y_3 \; .$$
One may introduce another representation, in terms of angular variables:
$$\begin{aligned} ct &= s \cosh \alpha \\ x &= s \sinh \alpha \cos \phi \sin \theta \\ y &= s \sinh \alpha \sin \phi \sin \theta \\ z &= s \sinh \alpha \cos \theta \; . \end{aligned}$$
where $\phi \in (0, 2\pi)$, $\theta \in (0, \pi)$, $s \geq 0$ and $\left\langle X, X \right\rangle \geq 0$.
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42. In practice one only excludes sequences of Brownian kicks that keep the Brownian particle close to the surface of the light cone for a long time. At the same time, it is unlikely that the average speed of a massive particle is close to the speed of light after a large number of kicks. A corresponding theory for spacelike events can be developed. However, requiring spacelike events to be physical would also require timelike events to be "unphysical" and vice-versa. In order words, we cannot use analytical continuity to pass from timelike to spacelike events in any physically meaningful way. A more detailed discussion can be found in [10,49].
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